Math 247A Lecture 12 Notes

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1 Calderón-Zygmund Decomposition and Bounds for the Vector-Valued Maximal Function

1.1 A Calderón-Zygmund decomposition

Lemma 1.1 (A Calderón-Zygmund decomposition)**.** *If $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$, then we can decompose $f = g + b$ such that*

1. $|g(x)| \leq \lambda$ *for almost every* $x \in \mathbb{R}^d$.
2. $\operatorname{supp} b$ *is a union of cubes whose interiors are pairwise disjoint and*

$$\lambda < \frac{1}{|Q_k|} \int_{Q_k} |b(x)| \, dx \leq 2^d \lambda.$$

3. $g = f[1 - \mathbb{1}_{\bigcup Q_k}]$.

Remark 1.1.

1. Interpolating between the first conclusion and $g \in L^1$, we get $g \in L^p$ for all $1 \leq p \leq \infty$.
2. $\sum_k |Q_k| \sim \frac{1}{\lambda} \sum_k \int_k \int_{Q_k} |b(y)| \, dy$, so $\sum_k |Q_k| \lesssim \frac{1}{\lambda} \|f\|_{L^1}$.

Remark 1.2. Modifying g further, we can ensure that $\int_{Q_k} b(y) \, dy = 0$ for all k. Indeed, let

$$g(x) = \begin{cases} f(x) & x \notin \bigcup_k Q_k \\ \frac{1}{|Q_k|} \int_{Q_k} f(y) \, dy & x \in Q_k. \end{cases}$$

Then for $x \in Q_k$,

$$b(x) = f(x) - \frac{1}{|Q_k|} \int_{Q_k} f(y) \, dy,$$

so

$$\int_{Q_k} b(x) \, dx = \int_{Q_k} f(x) \, dx - \int_{Q_k} f(y) \, dy = 0.$$

We lose a factor of 2 for the constant:

$$\frac{1}{|Q_k|}\int_{Q_k} |b(x)|\,dx \leq \frac{2}{|Q_k|}\int_{Q_k} |f(x)|\,dx \leq 2^{d+1}\lambda.$$

The price we have to pay is that $|g(x)| \leq 2^d\lambda$ instead of λ.

Proof. Decompose $\mathbb{R}^d$ into dyadic cubes $Q = [2^n k_1, 2^n(k_1+1)] \times \cdots \times [2^n k_d, 2^n(k_d+1)]$, where n is sufficiently large so that

$$\frac{1}{|Q|}\int_Q |f(y)|\,dy \leq \lambda$$

Fix such a Q and subdivide it into 2^d congruent cubes (cut each side in half). Let Q' denote one of the resulting children.

- If $\frac{1}{|Q'|}\int_{Q'} |f(y)|\,dy > \lambda$, stop and add Q' to the collection Q_k.
- If $\frac{1}{|Q'|}\int_{Q'} |f(y)|\,dy \leq \lambda$, then continue subdividing until (if ever) we are forced into case 1.

If we are in case 1, then

$$\lambda < \frac{1}{|Q'|}\int_{Q'} |f(y)|\,dy \leq \frac{2^d}{|Q|}\int_Q |f(y)|\,dy \leq 2^d\lambda.$$

It remains to show that $g = f[1 - \mathbb{1}_{\bigcup Q_k}]$ satisfies $|g| \leq \lambda$ a.e. Fix a Lebesgue point $x \notin \bigcup Q_k$ for f. Then

$$\left|\frac{1}{|Q|}\int_Q f(y)\ dy - f(x)\right| \leq \frac{1}{|Q|}\int_Q |f(y) - f(x)|\,dx$$

for any cube, we can inscribe a ball in side it and we can circumscribe a ball around it. Letting $r \sim \operatorname{diam}(Q)$,

$$\begin{aligned} &\lesssim \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y) - f(x)\,dx \\ &\xrightarrow{r\to\infty} 0 \end{aligned}$$

So

$$f(x) = \lim_{\substack{x \ni Q \\ \operatorname{diam} Q \to 0}} \frac{1}{|Q|}\int_Q f(y)\,dy,$$

and we get $|g(x)| = |f(x)| \leq \lambda$. ∎

1.2 Weak-type bound for the vector-valued maximal function

Recall that for $f : \mathbb{R}^d\ to \ell^2$ with $f = \{f_n\}_{n\geq 1}$, the **vector-valued maximal function** is

$$\overline{M}f(x) = \|\{Mf_n\}_{n\geq 1}\|_{\ell^2}.$$

Theorem 1.1.

1. *$\overline{M}$ is of weak-type $(1,1)$.*
2. *For $1 < p < \infty$, $\overline{M}$ is of strong type (p,p).*

Proof. Last time, we remarked that we need only prove part 1. Fix $f \in L^1$ and $\lambda > 0$. We want to show that

$$|\{x : \overline{M}f(x) > \lambda\}| \lesssim \frac{\|f\|_{L^1}}{\lambda}.$$

Decompose $f = g + b$ with $|g| \leq \lambda$ a.e., $\operatorname{supp} b = \bigcup_k Q_k$, and $\frac{1}{|Q_k|}\int_{Q_k} f|b(y)|\,dy \sim \lambda$. Then

$$|\{x : \overline{M}f(x) > \lambda\}| \leq |\{x : \overline{M}g(x) > \lambda/2\}| + |\{x : \overline{M}b(x) > \lambda/2\}|.$$

By Chebyshev,

$$|\{x : \overline{M}g(x) > \lambda/2\}| \lesssim \frac{\|\overline{M}g\|_2^2}{\lambda^2} \lesssim \frac{\|g\|_2^2}{\lambda^2} \lesssim \frac{\lambda\|g\|_{L^1}}{\lambda^2} \lesssim \frac{\|f\|_{L^1}}{\lambda}.$$

It is left to show that

$$|\{x : \overline{M}b(x) > \lambda/2\}| \lesssim \frac{\|f\|_{L^1}}{\lambda}.$$

We have

$$\sum_k |2Q_k| \leq 2^d \sum_k |Q_k| \sim \sum_k \frac{1}{\lambda}\int_{Q_k} |b(y)|\,dy \lesssim \frac{\|f\|_{L^1}}{\lambda}.$$

We have to show now that

$$|\{x \in [\bigcup(2Q_k)]^c : \overline{M}b(x) > \lambda/2\}| \lesssim \frac{\|f\|_{L^1}}{\lambda}.$$

For $x \notin \bigcup(2Q_k)$,

$$\begin{aligned} Mb_n(x) &= \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |b_n(y)|\,dy \\ &= \sup_{r>0} \frac{1}{|B(x,r)|}\sum_k \int_{B(x,r)\cap Q_k} |b_n(y)|\,dy \end{aligned}$$

If $B(x,r)\cap Q_k \neq \varnothing$, then $r > \ell(Q_k)/2$. So $Q_k \subseteq B(x, r + \sqrt{d}\ell(Q_k)) \subseteq B(x, r(1+2\sqrt{d}))$.

$$\leq_d \sup_{r>0} \frac{1}{|B(x,r(1+2\sqrt{d}))|}\sum_k \int_{B(x,r(1+2\sqrt{d}))} |b_n(y)|\,dy$$

$$\lesssim \sup_{r>0} \frac{1}{|B(x,r(1+2\sqrt{d}))|} \int_{B(x,r(1+2\sqrt{d}))} \sum_k \mathbb{1}_{Q_k}(z) \left(\frac{1}{|Q_k|} \int_{Q_k} |b_n(y)|\, dy \right) dz.$$

Let $b_n^{\text{avg}} = \sum \mathbb{1}_{Q_k} \frac{1}{|Q_k|} \int_{Q_k} |b_n(y)|\, dy$. Then we have

$$Mb_n(x) \lesssim Mb_n^{\text{avg}}(x).$$

Let $b^{\text{avg}} = \{b_n^{\text{avg}}\}_{n\geq 1}$. For $x \in Q_k$,

$$|b^{\text{avg}}(x)| = \|\{b_n^{\text{avg}}(x)\}_n\|_{\ell^2} \leq \frac{1}{|Q_k|} \int_{Q_k} |b(y)|\, dy \lesssim \lambda.$$

We also have

$$\|b^{\text{avg}}\|_{L^1} = \sum_k \int_{Q_k} |b(y)|\, dy \lesssim \|f\|_{L^1}.$$

By Chebyshev, since $\overline{M}b(x) \lesssim Mb^{\text{avg}}(x)$,

$$\begin{aligned} |\{x \in [\textstyle\bigcup(2Q_k)]^c : \overline{M}b(x) > \lambda/2\}| &\lesssim |\{x \in [\textstyle\bigcup(2Q_k)]^c : \overline{M}b^{\text{avg}} \gtrsim\}| \\ &\lesssim \frac{1}{\lambda^2} \|\overline{M}b^{\text{avg}}\||_{L^2}^2 \\ &\lesssim \frac{\|b^{\text{avg}}\|_{L^2}^2}{\lambda^2} \\ &\lesssim \frac{\|b^{\text{avg}}\|_{L^1}}{\lambda} \lesssim \frac{\|f\|_{L^1}}{\lambda}. \end{aligned}$$

□

Remark 1.3. One can replace ℓ^2 by ℓ^q for $1 < q \leq \infty$ for $f : \mathbb{R}^d\ toell^q$. Define

$$\overline{M}_q f(x) = \|\{Mf_n(x)\}_{n\geq 1}\|_{\ell^q}.$$

Then

1. $\overline{M}_q$ is of weak-type $(1,1)$.
2. $\overline{M}_q$ is of strong type (p,p) for all $1 < p < \infty$.

The proof is as in the case $q = 2$ if $1 < q < \infty$. The trivial estimate becomes that $\overline{M}_q : L^q \to L^q$ is bounded. If $q = \infty$,

$$\overline{M}_\infty f \leq M\|\{f_n\}_{n\geq 1}\|_{\ell^\infty}.$$

The estimates follow from the scalar case.

If $q = 1$, then these estimates fail. We will see an example next time.